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Summations for certain series containing the digamma function

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Abstract
Apparently new summations in terms of well-known special functions are deduced for hypergeometric-type series containing a digamma function as a factor. As a by-product of this investigation new reduction formulae for the Kampé de Fériet function $F^{0:2;1}_{2:1;0}[x, x]$ are obtained.

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1. Introduction

Although numerous series containing the digamma or psi function $\psi(z)$ (which is usually defined as the logarithmic derivative of the gamma function, i.e. $\psi(z) \equiv \Gamma'(z)/\Gamma(z)$) have been collected and recorded by Hansen [3, pp 360–366] many significant gaps still remain. Thus it is the purpose of this work to deduce several apparently new results among which are

$$\sum_{k=1}^{\infty} \frac{\psi(b+k)}{k!} x^k = \frac{x\,\mathrm{e}^x}{b} \left\{ {}_2F_2\left(\begin{matrix} 1 & , & 1 & ; \\ 2 & , & b+1 & ; \end{matrix} -x \right) + b\psi(b) \left(\frac{1-\mathrm{e}^{-x}}{x} \right) \right\}, \tag{1.1a}$$

$$\sum_{k=1}^{\infty} \frac{\psi(b+k)}{(b)_k} x^k = \frac{x\,\mathrm{e}^x}{b^2} \left\{ {}_2F_2\left(\begin{matrix} b & , & b & ; \\ b+1 & , & b+1 & ; \end{matrix} -x \right) + b\psi(b)\,{}_1F_1\left(\begin{matrix} b & ; \\ b+1 & ; \end{matrix} -x \right) \right\}, \tag{1.1b}$$

and for $|x| < 1$

$$\sum_{k=1}^{\infty} \psi(b+k) x^k = \frac{x}{1-x} \left\{ \psi(b) + \frac{1}{b}\,{}_2F_1\left(\begin{matrix} 1, b & ; \\ b+1 & ; \end{matrix} x \right) \right\} \tag{1.2}$$

(which may be rewritten also as equation (3.6)), where the ${}_pF_q$ are generalized hypergeometric functions, the Pochhammer symbol $(b)_k$ is defined by $(b)_k \equiv \Gamma(b+k)/\Gamma(b)$, and x and b are complex numbers such that $b \neq 0, -1, -2, \ldots$. Note that when $b = 1$ equations (1.1a) and (1.1b) coincide since $(1)_k = k!$ and ${}_1F_1(1; 2; -x) = (1 - \mathrm{e}^{-x})/x$. Note also that the confluent hypergeometric function in equation (1.1*b*) is essentially an incomplete gamma function since ${}_1F_1(b; b+1; -x) = bx^{-b}\gamma(b, x)$.

Sums of the type containing the digamma function considered herein occur often in mathematical physics and other applied areas especially when deriving asymptotic expansions and exact results for Mellin–Barnes (see e.g. [8, 9]) and other integrals. Thus, when possible it is advantageous to be able to recognize and express such sums in terms of known special functions in order to facilitate easier computation or further theoretical developments. Additional results will be discussed in section 5. In section 6 as a by-product of this investigation we deduce two new reduction formulae for the Kampé de Fériet function $F^{0:2;1}_{2:1;0}[x, x]$.

2. Transformation and reduction formulae

Essentially we shall derive equations (1.1) and (1.2) by using and exploiting appropriate connections between three results involving Kampé de Fériet functions (see e.g. [11] for an introduction to these functions) that have already been recorded earlier.

First, we note the transformation formula

$$F^{1:p;0}_{1:q;0}\left[\begin{matrix}\alpha & : & (a_p); & \text{——}; \\ \beta & : & (b_q); & \text{——};\end{matrix}\, x, y\right] = \mathrm{e}^y F^{0:p+1;1}_{1:q;0}\left[\begin{matrix}\text{——} & : & \alpha, (a_p) & ; & \beta-\alpha; \\ \beta & : & (b_q) & ; & \text{——};\end{matrix}\, x, -y\right], \tag{2.1}$$

where (a_p) denotes the sequence of parameters $a_1, a_2, \ldots, a_p$. This result is derived in [6, equation (3.6)] by essentially writing the Kampé de Fériet function on the left as a series indexed by $m \geqslant 0$ containing the confluent functions ${}_1F_1(\alpha+m; \beta+m; y)$ and then utilizing Kummer's first theorem ${}_1F_1(a; b; y) = \mathrm{e}^y {}_1F_1(b-a; b; -y)$ to obtain the right side of equation (2.1). Note that when $x = 0$ equation (2.1) reduces to Kummer's result with $a = \alpha$ and $b = \beta$.

Second, we shall need the reduction formula

$$F^{r:1;0}_{s:1;0}\left[\begin{matrix}(\alpha_r) & : & a; & \text{——}; \\ (\beta_s) & : & b; & \text{——};\end{matrix}\, x, -x\right] = {}_{r+1}F_{s+1}\left(\begin{matrix}(\alpha_r), b-a & ; \\ (\beta_s), b & ;\end{matrix}\, -x\right) \tag{2.2}$$

which is recorded in [11, p 31, equation (45)] in a slightly different form and can easily be shown to be a consequence of Gauss's summation theorem for the series ${}_2F_1(1)$.

Third, we shall utilize the reduction formula derived in [7, p 201]:

$$F^{p:2;1}_{q:1;0}\left[\begin{matrix}(\mu_p) & : & b-1, 1 & ; & 1 & ; \\ (\nu_q) & : & b & ; & \text{——} & ;\end{matrix}\, x, x\right] = (1-b)\psi(b-1)\,{}_{p+1}F_q(1, (\mu_p); (\nu_q); x)$$
$$- (1-b)\sum_{m=0}^{\infty}\psi(b+m)\frac{(\mu_p)_m}{(\nu_q)_m}x^m, \tag{2.3}$$

where $b \neq 0, -1, -2, \ldots, (\mu_p)_m \equiv (\mu_1)_m(\mu_2)_m \cdots (\mu_p)_m$ and when $p = 0$ the latter product is empty and understood to reduce to unity.

We mention that if only the upper (numerator) parameter $b-1$ in the Kampé de Fériet function in equation (2.3) is replaced by α, then a more general reduction formula may be obtained in terms of two generalized hypergeometric functions (cf [5, equation (9)]), but in this case the result just alluded to is not valid when $\alpha = b-1$.

In what follows we shall utilize two forms of a specialization of equation (2.3). In particular, set $p = 0$ and let $b \mapsto b+1$ thus giving

$$F^{0:2;1}_{q:1;0}\left[\begin{matrix}\text{——} & : & b, 1 & ; & 1 & ; \\ (\nu_q) & : & b+1 & ; & \text{——} & ;\end{matrix}\, x, x\right] = -b\psi(b)\,{}_1F_q\left(\begin{matrix}1 & ; \\ (\nu_q) & ;\end{matrix}\, x\right) + b\sum_{m=0}^{\infty}\frac{\psi(b+1+m)}{(\nu_q)_m}x^m$$

which may also be written as

$$F^{0:2;1}_{q:1;0}\left[\begin{matrix} \text{——} & : & b,1 & ; & 1 & ; \\ (\nu_q) & : & b+1 & ; & \text{——} & ; \end{matrix} x,x\right] = b\sum_{m=0}^{\infty}(\psi(b+1+m)-\psi(b))\frac{x^m}{(\nu_q)_m}.$$

In these two results replace m by $m-1$ and adjust the initial summation index accordingly. Thus since $(\alpha)_{m-1} = (\alpha-1)_m/(\alpha-1)$ we have

$$F^{0:2;1}_{q:1;0}\left[\begin{matrix} \text{——} & : & b,1 & ; & 1 & ; \\ (\nu_q) & : & b+1 & ; & \text{——} & ; \end{matrix} x,x\right] = -b\psi(b)\,{}_1F_q\left(\begin{matrix} 1 & ; \\ (\nu_q) & ; \end{matrix} x\right)$$
$$+\frac{b}{x}\sum_{m=1}^{\infty}\psi(b+m)\left\{\prod_{i=1}^{q}\frac{\nu_i-1}{(\nu_i-1)_m}\right\}x^m \qquad (2.4a)$$

and

$$F^{0:2;1}_{q:1;0}\left[\begin{matrix} \text{——} & : & b,1 & ; & 1 & ; \\ (\nu_q) & : & b+1 & ; & \text{——} & ; \end{matrix} x,x\right] = \frac{b}{x}\sum_{m=1}^{\infty}(\psi(b+m)-\psi(b))\left\{\prod_{i=1}^{q}\frac{\nu_i-1}{(\nu_i-1)_m}\right\}x^m \qquad (2.4b)$$

where in the latter the index may be started at $m=0$ since it provides no contribution.

3. Summation of the series in equations (1.1) and (1.2)

In equation (2.1) let $p=q=1$ and $y=-x$. Thus dispensing with the parametric subscripts (here and in what follows) gives

$$F^{1:1;0}_{1:1;0}\left[\begin{matrix} \alpha & : & a; & \text{——}; \\ \beta & : & b; & \text{——}; \end{matrix} x,-x\right] = \mathrm{e}^{-x}F^{0:2;1}_{1:1;0}\left[\begin{matrix} \text{——} & : & \alpha,a & ; & \beta-\alpha & ; \\ \beta & : & b & ; & \text{——} & ; \end{matrix} x,x\right].$$

And in equation (2.2) letting $r=s=1$ yields

$$F^{1:1;0}_{1:1;0}\left[\begin{matrix} \alpha & : & a; & \text{——}; \\ \beta & : & b; & \text{——}; \end{matrix} x,-x\right] = {}_2F_2\left(\begin{matrix} \alpha,b-a & ; \\ \beta,b & ; \end{matrix} -x\right).$$

Thus we have the reduction formula

$$F^{0:2;1}_{1:1;0}\left[\begin{matrix} \text{——} & : & \alpha,a & ; & \beta-\alpha; \\ \beta & : & b & ; & \text{——}; \end{matrix} x,x\right] = \mathrm{e}^{x}\,{}_2F_2\left(\begin{matrix} \alpha,b-a & ; \\ \beta,b & ; \end{matrix} -x\right)$$

which when specialized with $\alpha=c$, and $\beta=c+1$ gives

$$F^{0:2;1}_{1:1;0}\left[\begin{matrix} \text{——} : & a,c & ; & 1 & ; \\ c+1 : & b & ; & \text{——} & ; \end{matrix} x,x\right] = \mathrm{e}^{x}\,{}_2F_2\left(\begin{matrix} c,b-a & ; \\ c+1,b & ; \end{matrix} -x\right). \qquad (3.1)$$

In equation (3.1) letting first $b\mapsto b+1$ and then setting $c=1$, $a=b$ provides

$$F^{0:2;1}_{1:1;0}\left[\begin{matrix} \text{——} & : & b,1 & ; & 1 & ; \\ 2 & : & b+1 & ; & \text{——} & ; \end{matrix} x,x\right] = \mathrm{e}^{x}\,{}_2F_2\left(\begin{matrix} 1,1 & ; \\ 2,b+1 & ; \end{matrix} -x\right). \qquad (3.2a)$$

But from equations (2.4a) and (2.4b) respectively upon setting $q=1$, and $\nu=2$ we have also

$$F^{0:2;1}_{1:1;0}\left[\begin{matrix} \text{——} & : & b,1 & ; & 1 & ; \\ 2 & : & b+1 & ; & \text{——} & ; \end{matrix} x,x\right] = -b\psi(b)\,{}_1F_1(1;2;x)+\frac{b}{x}\sum_{m=1}^{\infty}\frac{\psi(b+m)}{m!}x^m \qquad (3.2b)$$

and

$$F^{0:2;1}_{1:1;0}\left[\begin{matrix} \text{——} & : & b,1 & ; & 1 & ; \\ 2 & : & b+1 & ; & \text{——} & ; \end{matrix} x,x\right] = \frac{b}{x}\sum_{m=1}^{\infty}(\psi(b+m)-\psi(b))\frac{x^m}{m!}. \qquad (3.2c)$$

Thus from equations (3.2) upon noting that ${}_1F_1(1;2;x) = (\mathrm{e}^x - 1)/x$ we have

$$\sum_{m=1}^{\infty} \frac{\psi(b+m)}{m!} x^m = \frac{x\,\mathrm{e}^x}{b}\,{}_2F_2\left(\begin{matrix}1 & , & 1 & ; \\ 2 & , & b+1 & ;\end{matrix} -x\right) + \psi(b)(\mathrm{e}^x - 1)$$

which may be rewritten as equation (1.1a) and the more elegant result

$$\sum_{m=1}^{\infty} [\psi(b+m) - \psi(b)] \frac{x^m}{m!} = \frac{x\,\mathrm{e}^x}{b}\,{}_2F_2\left(\begin{matrix}1 & , & 1 & ; \\ 2 & , & b+1 & ;\end{matrix} -x\right). \tag{3.3}$$

Now in equation (3.1) letting first $b \mapsto b+1$ and then setting $a = 1$, $c = b$ provides

$$F^{0:2;1}_{1:1;0}\left[\begin{matrix}\text{——} & : & b, 1 & ; & 1 & ; \\ b+1 & : & b+1 & ; & \text{——} & ;\end{matrix} x, x\right] = \mathrm{e}^x\,{}_2F_2\left(\begin{matrix}b & , & b & ; \\ b+1 & , & b+1 & ;\end{matrix} -x\right). \tag{3.4a}$$

And in equation (2.4a) set $q = 1$, and $\nu = b + 1$ thus giving also

$$F^{0:2;1}_{1:1;0}\left[\begin{matrix}\text{——} & : & b, 1 & ; & 1 & ; \\ b+1 & : & b+1 & ; & \text{——} & ;\end{matrix} x, x\right] = -b\psi(b)\,{}_1F_1(1; b+1; x) + \frac{b^2}{x} \sum_{m=1}^{\infty} \frac{\psi(b+m)}{(b)_m} x^m. \tag{3.4b}$$

Now equating the right sides of equations (3.4), and rearranging terms yields

$$\frac{b^2}{x} \sum_{m=1}^{\infty} \frac{\psi(b+m)}{(b)_m} x^m = \mathrm{e}^x\,{}_2F_2\left(\begin{matrix}b & , & b & ; \\ b+1 & , & b+1 & ;\end{matrix} -x\right) + b\psi(b)\,{}_1F_1(1; b+1; x).$$

Since by Kummer's first theorem ${}_1F_1(1; b+1; x) = \mathrm{e}^x\,{}_1F_1(b; b+1; -x)$, we deduce equation (1.1b).

Finally, to derive equation (1.2) in equation (2.4a) let $q = 0$ thus giving

$${}_2F_1(1, b; b+1; x)\,{}_1F_0(1; -; x) = -b\psi(b)\,{}_1F_0(1; -; x) + \frac{b}{x} \sum_{m=1}^{\infty} \psi(b+m) x^m. \tag{3.5}$$

Noting that ${}_1F_0(1; -; x) = (1-x)^{-1}$, and rearranging terms then yields

$$\frac{b}{x} \sum_{m=1}^{\infty} \psi(b+m) x^m = \frac{1}{1-x} [b\psi(b) + {}_2F_1(1, b; b+1; x)].$$

Now multiplying both sides of the latter equation by x/b we have equation (1.2). Note that since ${}_1F_0(1; -; x)$ in equation (3.5) does not converge on the unit circle $|x| = 1$, equation (1.2) is valid only when $|x| < 1$.

Another derivation of equation (1.2) is given as follows. Starting with the functional relation for the digamma function

$$\psi(z+1) = \psi(z) + \frac{1}{z}$$

let $z = b + k$, multiply both sides of the latter by x^k, and sum over $k \geqslant 0$ thus giving

$$\sum_{k=0}^{\infty} \psi(1+b+k) x^k = \sum_{k=0}^{\infty} \psi(b+k) x^k + \sum_{k=0}^{\infty} \frac{x^k}{b+k}.$$

Now adjusting the summation index on the left and observing that the second sum on the right is equal to ${}_2F_1(1, b; b+1; x)/b$, where $b \neq 0, -1, -2, \ldots$ and $|x| < 1$ for convergence of Gauss's function ${}_2F_1(x)$ we have

$$\frac{1}{x}\sum_{k=1}^{\infty}\psi(b+k)x^k = \psi(b) + \sum_{k=1}^{\infty}\psi(b+k)x^k + \frac{1}{b}{}_2F_1\left(\begin{matrix}1, b & ; \\ b+1 & ;\end{matrix}\, x\right).$$

Solving for the k-summation then yields equation (1.2) which may also be written more elegantly as

$$\sum_{k=1}^{\infty}[\psi(b+k) - \psi(b)]x^k = \frac{1}{b}\frac{x}{1-x}{}_2F_1\left(\begin{matrix}1, b & ; \\ b+1 & ;\end{matrix}\, x\right), \tag{3.6}$$

where $b \neq 0, -1, -2, \ldots$ and $|x| < 1$. Note that equation (3.6) may be obtained immediately from equation (2.4b) upon setting $q = 0$.

4. The specialization $b = 1$

We mentioned in section 1 that when $b = 1$, equations (1.1a) and (1.1b) coincide. Furthermore, in this case we have from either of equations (1.1) or (3.3)

$$\sum_{k=1}^{\infty}[\psi(k+1) - \psi(1)]\frac{x^k}{k!} = x\,\mathrm{e}^x{}_2F_2\left(\begin{matrix}1, 1 & ; \\ 2, 2 & ;\end{matrix}\, -x\right), \tag{4.1}$$

where x is an arbitrary complex number. An alternative elementary derivation of equation (4.1) (which is given in a much different form) is outlined in [1, p 459, example 16].

However, if x is restricted to nonzero real values since for $x > 0$

$$x\,{}_2F_2\left(\begin{matrix}1, 1 & ; \\ 2, 2 & ;\end{matrix}\, -x\right) = \gamma + \ln x - \mathrm{Ei}(-x)$$

and

$$x\,{}_2F_2\left(\begin{matrix}1, 1 & ; \\ 2, 2 & ;\end{matrix}\, x\right) = \mathrm{Ei}(x) - \gamma - \ln x,$$

where $\mathrm{Ei}(x)$ is the exponential-integral function (cf [2, section 8.214]) and $\gamma = -\psi(1)$ is Euler's constant, we have from equation (4.1) for $x > 0$

$$\sum_{k=1}^{\infty}[\psi(k+1) - \psi(1)]\frac{x^k}{k!} = \mathrm{e}^x[\gamma + \ln x - \mathrm{Ei}(-x)] \tag{4.2a}$$

and

$$\sum_{k=1}^{\infty}[\psi(k+1) - \psi(1)]\frac{(-x)^k}{k!} = \mathrm{e}^{-x}[\gamma + \ln x - \mathrm{Ei}(x)]. \tag{4.2b}$$

Equations (4.2) are recorded by Hansen (see [3, p 363, equations (55.7.1) and (55.7.2)], where other references are also provided), but the restriction on the variable x is ambiguous.

Equations (4.2) may be written in simplified forms as

$$\sum_{k=0}^{\infty}\psi(k+1)\frac{x^k}{k!} = \mathrm{e}^x[\ln x - \mathrm{Ei}(-x)]$$

and

$$\sum_{k=0}^{\infty}\psi(k+1)\frac{(-x)^k}{k!} = \mathrm{e}^{-x}[\ln x - \mathrm{Ei}(x)],$$

where $x > 0$.

The specialization $b = 1$ in equation (1.2) gives for $|x| < 1$

$$\sum_{k=1}^{\infty} \psi(k+1)x^k = \frac{x}{1-x}[\psi(1) + {}_2F_1(1, 1; 2; x)].$$

Upon multiplying both sides of this by x, adjusting the summation index, and noting that ${}_2F_1(1, 1; 2; x) = -x^{-1}\ln(1-x)$ we have

$$\sum_{k=2}^{\infty} \psi(k)x^k = \frac{x^2}{1-x}\psi(1) - \frac{x\ln(1-x)}{1-x}.$$

And since

$$\sum_{k=2}^{\infty} x^k = \frac{x^2}{1-x}$$

we see that

$$\sum_{k=2}^{\infty} [\psi(k) - \psi(1)]x^k = \frac{x\ln(1-x)}{x-1}, \qquad |x| < 1$$

which is recorded in [3, p 361, equation (55.3.1)].

5. Additional results

Upon letting $b \mapsto b+1$ we see that equation (2.3) may be written more elegantly as

$$\sum_{m=0}^{\infty} [\psi(b+1+m) - \psi(b)]\frac{(\mu_p)_m}{(\nu_q)_m}x^m = \frac{1}{b}F^{p:2;1}_{q:1;0}\left[\begin{matrix}(\mu_p) & : & b, 1 & ; & 1 & ; \\ (\nu_q) & : & b+1 & ; & \text{---} & ;\end{matrix}\, x, x\right], \tag{5.1}$$

where $b \neq 0, -1, -2, \ldots$. The leftmost sum of the latter is the most general generalized hypergeometric-type sum containing the single digamma function $\psi(b+1+m)$ as a factor; and equation (5.1) shows that these sums are truly hypergeometric in the sense that they may be written essentially as a double generalized hypergeometric (i.e. a Kampé de Fériet) function in two (equal) variables. Moreover, when the latter function is capable of reduction to a known hypergeometric or special function in one variable, equation (5.1) becomes especially useful. We have already exploited this feature in section 3 and we shall provide additional applications in what follows.

In equation (2.4*b*) let $q = 2$ and set $\nu_1 = 2$, $\nu_2 = b+1$. Thus upon multiplying by x/b^2 we obtain

$$\sum_{m=1}^{\infty} [\psi(b+m) - \psi(b)]\frac{x^m}{(b)_m m!} = \frac{x}{b^2}F^{0:2;1}_{2:1;0}\left[\begin{matrix}\text{---} & : & b, 1 & ; & 1 & ; \\ 2, b+1 & : & b+1 & ; & \text{---} & ;\end{matrix}\, x, x\right]. \tag{5.2}$$

In [7, equation (6.12)] we obtained a reduction formula for $F^{0:2;1}_{2:1;0}[x, x]$ that with various renamings of the parameters and variables may be written as

$$\begin{aligned} F^{0:2;1}_{2:1;0}&\left[\begin{matrix}\text{---} & : & \alpha, \beta & ; & 1 & ; \\ \alpha+1, \beta+1 & : & \gamma & ; & \text{---} & ;\end{matrix}\, x, x\right] \\ &= \frac{\alpha}{\alpha-\beta}\,{}_0F_1\left(\begin{matrix}\text{---} & ; \\ 1+\alpha-\beta & ;\end{matrix}\, x\right){}_3F_4\left(\begin{matrix}\beta, \frac{\gamma+\beta-\alpha}{2}, \frac{1+\gamma+\beta-\alpha}{2} & ; \\ \gamma, \beta+1, 1+\beta-\alpha, \gamma+\beta-\alpha & ;\end{matrix}\, 4x\right) \\ &\quad + \frac{\beta}{\beta-\alpha}\,{}_0F_1\left(\begin{matrix}\text{---} & ; \\ 1+\beta-\alpha & ;\end{matrix}\, x\right){}_3F_4\left(\begin{matrix}\alpha, \frac{\gamma+\alpha-\beta}{2}, \frac{1+\gamma+\alpha-\beta}{2} & ; \\ \gamma, \alpha+1, 1+\alpha-\beta, \gamma+\alpha-\beta & ;\end{matrix}\, 4x\right). \end{aligned}$$

In this set $\beta = 1, \alpha = b, \gamma = b+1$ which yields

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix} \text{---} & : & b, 1 & ; & 1 & ; \\ 2, b+1 & : & b+1 & ; & \text{---} & ; \end{matrix} x, x\right]$$
$$= \frac{b}{b-1}\,{}_0F_1\left(\begin{matrix}\text{---}\\ b\end{matrix};x\right){}_3F_4\left(\begin{matrix}1,1,3/2\\ 1+b,2,2-b,2\end{matrix};4x\right)$$
$$+\frac{1}{1-b}\,{}_0F_1\left(\begin{matrix}\text{---}\\ 2-b\end{matrix};x\right){}_3F_4\left(\begin{matrix}b,b,\frac{1+2b}{2}\\ 1+b,1+b,b,2b\end{matrix};4x\right). \tag{5.3}$$

The latter hypergeometric function reduces to one of lower order and so from equations (5.2) and (5.3) we have

$$\sum_{m=1}^{\infty}[\psi(b+m)-\psi(b)]\frac{x^m}{(b)_m m!} = \frac{x}{b(b-1)}\left[{}_0F_1\left(\begin{matrix}\text{---}\\ b\end{matrix};x\right){}_3F_4\left(\begin{matrix}1,1,3/2\\ 2,2,1+b,2-b\end{matrix};4x\right)\right.$$
$$\left.-\frac{1}{b}\,{}_0F_1\left(\begin{matrix}\text{---}\\ 2-b\end{matrix};x\right){}_2F_3\left(\begin{matrix}b,\frac{1}{2}+b\\ 2b,1+b,1+b\end{matrix};4x\right)\right] \tag{5.4}$$

where b is not an integer (positive, negative or zero).

For example, if we set $b = 1/2$ and let $x \mapsto -x^2/4$ in equation (5.4), since there is no contribution to the sum from $m = 0$ we obtain

$$\sum_{m=0}^{\infty}\left[\psi\left(\frac{1}{2}+m\right)-\psi\left(\frac{1}{2}\right)\right]\frac{(-x^2/4)^m}{(1/2)_m m!} = x^2\left[{}_0F_1\left(\begin{matrix}\text{---}\\ 1/2\end{matrix};-\frac{x^2}{4}\right){}_2F_3\right.$$
$$\left.\times\left(\begin{matrix}1,1\\ 2,2,3/2\end{matrix};-x^2\right)-2\,{}_0F_1\left(\begin{matrix}\text{---}\\ 3/2\end{matrix};-\frac{x^2}{4}\right){}_1F_2\left(\begin{matrix}1/2\\ 3/2,3/2\end{matrix};-x^2\right)\right]. \tag{5.5a}$$

Noting that $\psi(1/2) = -\gamma - 2\ln 2$ and observing that the four hypergeometric functions in the order of their appearance may be written essentially as a cosine, integral cosine, sine, and integral sine (see e.g. [10, section 7] for the exact pertinent connecting formulae) we obtain from the latter after simplification

$$\sum_{m=0}^{\infty}\psi\left(\frac{1}{2}+m\right)\frac{(-x^2/4)^m}{(1/2)_m m!} = \cos x\left[\ln\left(\frac{x}{2}\right)-\mathrm{ci}(2x)\right]-\sin x\left[\frac{\pi}{2}+\mathrm{si}(2x)\right], \tag{5.5b}$$

where the integral sine $\mathrm{si}(x)$ and cosine $\mathrm{ci}(x)$ are defined respectively by

$$\mathrm{si}(x) \equiv -\int_x^{\infty}\frac{\sin t}{t}\,\mathrm{d}t, \qquad \mathrm{ci}(x) \equiv -\int_x^{\infty}\frac{\cos t}{t}\,\mathrm{d}t.$$

The result given by equation (5.4) is not valid for positive integers b; and so in this case we shall derive a representation for the sum $\sum_{m=0}^{\infty}\psi(b+m)x^m/(b)_m m!$ by exploiting the result (cf [3, p 363, equation (55.7.11)])

$$\sum_{m=0}^{\infty}\psi(\nu+m)\frac{x^m}{(\nu)_m m!} = \Gamma(\nu)x^{\frac{1-\nu}{2}}\left[\frac{1}{2}I_{\nu-1}(2\sqrt{x})\ln x - \frac{\partial}{\partial\nu}I_{\nu-1}(2\sqrt{x})\right], \tag{5.6}$$

where $I_\nu(z)$ is the modified Bessel function.

It is known for nonnegative integers n (see e.g. [4, p 71, section 3.2.3]) that

$$\left.\frac{\partial I_\nu(z)}{\partial\nu}\right|_{\nu=n} = (-1)^n\left[-K_n(z)+\frac{1}{2}n!\sum_{k=0}^{n-1}\frac{(-1)^k(2/z)^{n-k}}{(n-k)k!}I_k(z)\right], \tag{5.7}$$

where $K_\nu(z)$ is the Macdonald function (or Bessel function of imaginary argument), and the latter finite sum vanishes when $n = 0$. Since

$$I_{\nu-1}(z) = \frac{\mathrm{d}}{\mathrm{d}z} I_\nu(z) + \frac{\nu}{z} I_\nu(z), \tag{5.8}$$

it is evident that

$$\frac{\partial}{\partial\nu} I_{\nu-1}(z)\bigg|_{\nu=n} = \frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial}{\partial\nu} I_\nu(z)\bigg|_{\nu=n} + \frac{n}{z}\frac{\partial}{\partial\nu} I_\nu(z)\bigg|_{\nu=n} + \frac{I_n(z)}{z}. \tag{5.9}$$

Moreover, since

$$\frac{\mathrm{d}}{\mathrm{d}z} K_\nu(z) = -K_{\nu-1}(z) - \frac{\nu}{z} K_\nu(z)$$

and from equation (5.8)

$$\frac{\mathrm{d}}{\mathrm{d}z} I_\nu(z) = I_{\nu-1}(z) - \frac{\nu}{z} I_\nu(z),$$

differentiating both members of equation (5.7) with respect to z yields

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial}{\partial\nu} I_\nu(z)\bigg|_{\nu=n} = (-1)^n \left[K_{n-1}(z) + \frac{n}{z}K_n(z) - \frac{n!}{2}\sum_{k=0}^{n-1}(-1)^k \frac{(2/z)^{n-k}}{k!}\frac{I_k(z)}{z} \right.$$
$$\left. + \frac{n!}{2}\sum_{k=0}^{n-1}\frac{(-1)^k(2/z)^{n-k}}{(n-k)k!}\left(I_{k-1}(z) - \frac{k}{z}I_k(z)\right)\right].$$

Thus, by using the latter together with equation (5.7) we have from equation (5.9) after simplification the result

$$\frac{\partial}{\partial\nu} I_{\nu-1}(z)\bigg|_{\nu=n} = \frac{I_n(z)}{z} + (-1)^n\left[K_{n-1}(z) + \frac{1}{2}n!\sum_{k=0}^{n-1}\frac{(-1)^k(2/z)^{n-k}}{(n-k)k!}I_{k-1}(z)\right], \tag{5.10}$$

where $n = 0, 1, 2, \ldots$. Finally, equations (5.10) with $z = 2\sqrt{x}$ and (5.6) with $\nu = n$ yield

$$\sum_{m=0}^{\infty}\psi(n+m)\frac{x^m}{(n)_m m!} = (n-1)!\,x^{\frac{1-n}{2}}\left\{\frac{1}{2}I_{n-1}(2\sqrt{x})\ln x - \frac{I_n(2\sqrt{x})}{2\sqrt{x}}\right.$$
$$\left. + (-1)^{n-1}\left[K_{n-1}(2\sqrt{x}) + \frac{1}{2}n!\sum_{k=0}^{n-1}\frac{(-1)^k x^{\frac{k-n}{2}}}{(n-k)k!}I_{k-1}(2\sqrt{x})\right]\right\}, \tag{5.11}$$

where $n = 1, 2, 3, \ldots$.

In particular, when $n = 1$ equation (5.11) provides (upon nothing that $I_{-n}(z) = I_n(z)$ for integers n)

$$\sum_{m=0}^{\infty}\psi(1+m)\frac{x^m}{m!^2} = \frac{1}{2}I_0(2\sqrt{x})\ln x + K_0(2\sqrt{x}) \tag{5.12}$$

which is a well-known result (cf [3, p 363, equation (55.7.7)]).

Next, we shall deduce a representation for $\sum_{k=0}^{\infty}\psi(1+k)x^k/(n)_k k!$ for positive integers n. In [8, p 253, Ex. 8.3] Olver records an expansion for the Macdonald function $K_n(z)$ for n a nonnegative integer:

$$K_n(z) = \frac{1}{2}\left(\frac{2}{z}\right)^n\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{-z^2}{4}\right)^k - (-1)^n\ln\left(\frac{z}{2}\right)I_n(z)$$
$$+ (-1)^n\frac{1}{2}\left(\frac{z}{2}\right)^n\sum_{k=0}^{\infty}[\psi(1+k) + \psi(1+n+k)]\frac{(z^2/4)^k}{k!(n+k)!}, \tag{5.13}$$

where the finite k-summation vanishes when $n = 0$. A few elementary manipulations reveal that this sum may be written in closed form as

$$\sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{-z^2}{4} \right)^k = \frac{(-z^2/4)^{n-1}}{\Gamma(n)} {}_3F_0(1, 1, 1-n; -; 4/z^2).$$

Thus rearranging the terms of equation (5.13), letting $n \mapsto n-1$, setting $z = 2\sqrt{x}$, and observing that $(n+k-1)! = (n)_k \Gamma(n)$ gives

$$\sum_{k=0}^{\infty} \frac{\psi(1+k)x^k}{(n)_k k!} + \sum_{k=0}^{\infty} \frac{\psi(n+k)x^k}{(n)_k k!} = \frac{n-1}{x} {}_3F_0(1, 1, 2-n; -; 1/x) + (n-1)! x^{\frac{1-n}{2}} [I_{n-1}(2\sqrt{x}) \ln x + 2(-1)^{n-1} K_{n-1}(2\sqrt{x})], \tag{5.14}$$

where $n = 1, 2, 3, \ldots$.

Finally, employing equation (5.11) we obtain for positive integers n

$$\begin{aligned}\sum_{k=0}^{\infty} \psi(1+k) \frac{x^k}{(n)_k k!} &= \frac{n-1}{x} {}_3F_0(1, 1, 2-n; -; 1/x) \\ &\quad + (n-1)! x^{\frac{1-n}{2}} \left\{ \frac{1}{2} I_{n-1}(2\sqrt{x}) \ln x + \frac{I_n(2\sqrt{x})}{2\sqrt{x})} \right. \\ &\quad \left. + (-1)^{n-1} \left[K_{n-1}(2\sqrt{x}) - \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{(-1)^k x^{\frac{k-n}{2}}}{(n-k)k!} I_{k-1}(2\sqrt{x}) \right] \right\}. \end{aligned} \tag{5.15}$$

Note that when $n = 1$, equations (5.14) and (5.15) reduce to equation (5.12).

6. Two reduction formulae

We may use equations (5.1) and (5.15) to obtain a heretofore unavailable reduction formula for the Kampé de Fériet function $F_{2:1;0}^{0:2;1}[x, x]$. In equation (2.4b) let $q = 2$ and set $\nu_1 = 2$, $\nu_2 = n+1$, $b = 1$ thus giving

$$\frac{n}{x} \sum_{m=1}^{\infty} [\psi(1+m) - \psi(1)] \frac{x^m}{(n)_m m!} = F_{2:1;0}^{0:2;1} \left[\begin{matrix} \text{——} & : & 1, 1 & ; & 1 & ; \\ 2, n+1 & : & 2 & ; & \text{——} & ; \end{matrix} x, x \right].$$

The left side of the latter may be written as

$$\frac{n}{x} \sum_{m=1}^{\infty} [\psi(1+m) + \gamma] \frac{x^m}{(n)_m m!} = \frac{n}{x} \sum_{m=0}^{\infty} [\psi(1+m) + \gamma] \frac{x^m}{(n)_m m!}$$

since $\psi(1) = -\gamma$. And so noting that

$${}_0F_1(-; n; x) = (n-1)! x^{\frac{1-n}{2}} I_{n-1}(2\sqrt{x}) \tag{6.1}$$

we deduce

$$\begin{aligned} F_{2:1;0}^{0:2;1} &\left[\begin{matrix} \text{——} & : & 1, 1 & ; & 1 & ; \\ 2, n+1 & : & 2 & ; & \text{——} & ; \end{matrix} x, x \right] \\ &= \frac{n}{x} \left\{ \gamma (n-1)! x^{\frac{1-n}{2}} I_{n-1}(2\sqrt{x}) + \sum_{m=0}^{\infty} \psi(1+m) \frac{x^m}{(n)_m m!} \right\}, \end{aligned} \tag{6.2}$$

where for positive integers n the summation is given by equation (5.15). Thus equation (6.2) provides the required reduction formula. When $n = 1$ this reduces via equation (5.12) to

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix} \text{——} & : & 1,1 & ; & 1 & ; \\ 2,2 & : & 2 & ; & \text{——} & ; \end{matrix} x, x\right] = \frac{1}{x}\left[\left(\gamma + \frac{1}{2}\ln x\right) I_0(2\sqrt{x}) + K_0(2\sqrt{x})\right]. \quad (6.3)$$

Another reduction formula may be obtained by employing equations (5.2) and (5.11) the former of which provides for $b = n$ a positive integer

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix} \text{——} & : & n,1 & ; & 1 & ; \\ 2,n+1 & : & n+1 & ; & \text{——} & ; \end{matrix} x, x\right] = \frac{n^2}{x}\sum_{m=0}^{\infty}[\psi(n+m) - \psi(n)]\frac{x^m}{(n)_m m!},$$

since the contribution to the sum from $m = 0$ is obviously zero. Thus using equation (6.1) we have

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix} \text{——} & : & n,1 & ; & 1 & ; \\ 2,n+1 & : & n+1 & ; & \text{——} & ; \end{matrix} x, x\right]$$
$$= \frac{n^2}{x}\left\{\sum_{m=0}^{\infty}\psi(n+m)\frac{x^m}{(n)_m m!} - \psi(n)(n-1)!x^{\frac{1-n}{2}} I_{n-1}(2\sqrt{x})\right\}, \quad (6.4)$$

where the summation is given by equation (5.11). When $n = 1$ equation (6.4) also reduces to equation (6.3). Moreover, the reduction formula given by equation (6.4) provides the analogue of equation (5.3) when b is a positive integer n.

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